Iterated forcing with side conditions

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Properness

(Shelah) A forcing notion $\mathcal{P}$ is *proper* iff for every cardinal $\theta > |\mathcal{P}|$, every countable $N \prec H(\theta)$ such that $\mathcal{P} \in N$ and every $p \in N \cap \mathcal{P}$ there is some $q \leq p$ such that

$$q \Vdash_{\mathcal{P}} D \cap \dot{G} \cap N \neq \emptyset$$

for every dense $D \subseteq \mathcal{P}$ such that $D \in N$.

We say that q is $(N, \mathcal{P})$–*generic*.

Note: $\mathcal{P}$ is proper iff the above holds for some $\theta > |\mathcal{P}|$.

Proper forcing is nice:

- Proper forcing notions preserve ω_1.
- Properness is preserved under countable support (CS) iterations.

Hence, granted the existence of a supercompact cardinal, one can build a model of PFA, the forcing axiom for proper forcings relative to collection of $\aleph_1$–many dense series (Baumgartner).

PFA: For every proper $\mathcal{P}$ and for every collection $\{D_i : i < \omega_1\}$ of dense subsets of $\mathcal{P}$ there is a filter $G \subseteq \mathcal{P}$ such that $G \cap D_i \neq \emptyset$ for all i.

PFA has many consequences. One of them is $2^{\aleph_0} = \aleph_2$.

Problem: Force some consequence of PFA or, for that matter, something we can force by iterating non–c.c.c. proper forcing, together with $2^{\aleph_0} > \aleph_2$.

Countable support iterations won't do. In fact, at stages of uncountable cofinality we are adding generics, over all previous models, for $\mathrm{Add}(1, \omega_1)$ (= adding a Cohen subset of ω_1); in particular we are collapsing the continuum of all those previous models to $\aleph_1$. Hence, in the final model necessarily $2^{\aleph_0} \leq \aleph_2$.

Bigger support won't work either: The preservation lemma for properness doesn't work in this context.

Finite support iterations won't work either; in fact, any finite support ω–length iteration of non–c.c.c. forcings collapses ω_1.

Side conditions

Rough idea: We're interested in forcing with a non-proper $\mathcal{P}$, and we would really like it to be proper. We can look at some similar forcing $\mathcal{P}^*$ *which incorporates countable models* as side conditions and is thereby proper.

First example perhaps Baumgartner's forcing for adding a club of ω_1 with finite condition.

Method made explicit in work of Todorčević from the 1980's.

Typical examples: Conditions in $\mathcal{P}^*$ are pairs of the form $(w, \mathcal{N})$, where

- w is the *working part* (adding the object we are ultimately interested in).
- $\mathcal{N}$ is a finite $\in$–chain (i.e., can be ordered as $(N_i)_{i<n}$ with $N_i \in N_{i+1}$ for all i) of elementary submodels of some suitable $H(\chi)$ containing all relevant objects.
- w is "generic for all members of $\mathcal{N}$".

Extension: $(w_1, \mathcal{N}_1) \leq (w_0, \mathcal{N}_0)$ iff

- w_1 extends w_0 (in some natural way), and
- $\mathcal{N}_0 \subseteq \mathcal{N}_1$.

Typical proof of properness:

- Start with $(w, \mathcal{N}) \in N$, N countable, $N \prec H(\theta)$ for large enough θ.
- **Add** $N \cap H(\chi)$ **to** $(w, \mathcal{N})$. That is, build $(\bar{w}, \mathcal{N} \cup \{N \cap H(\chi)\})$, where $\bar{w}$ is perhaps some extension of w.
- Prove that $(\bar{w}, \mathcal{N} \cup \{N \cap H(\chi)\})$ is $(N, \mathcal{P}^*)$–generic.

Example: Measuring one club–sequence by finite conditions.

Weak Club Guessing at ω_1 (WCG):
There is a ladder system $(C_\delta \,:\, \delta \in \mathrm{Lim}(\omega_1))$ (i.e., for all δ, $C_\delta \subseteq \delta$ is cofinal in δ and of order type ω) such that for every club $D \subseteq \omega_1$ there is some δ such that $|D \cap C_\delta| = \aleph_0$.

WCG is a very weak version of Jensen's $\diamondsuit$.

Killing one instance of WCG:

Let $\vec{C} = (C_\delta \,:\, \delta \in \mathrm{Lim}(\omega_1))$ ladder system. Let $\mathcal{P}_{\vec{C}}$ be as follows: Conditions are pairs (f, b) such that

(1) $f \subseteq \omega_1 \times \mathrm{Lim}(\omega_1)$ is a finite function that can be extended to a strictly increasing and continuous function $F : \omega_1 \longrightarrow \mathrm{Lim}(\omega_1)$.

(2) $\mathrm{dom}(b) = \mathrm{dom}(f)$ and $b(\xi) < f(\xi)$ for each $\xi \in \mathrm{dom}(b)$.

(3) For each $\xi \in \mathrm{dom}(b)$, $C_{f(\xi)} \cap \mathrm{range}(f \restriction \xi) \subseteq b(\xi)$.

Extension: $(f_1, b_1) \leq (f_0, b_0)$ iff

- $f_0 \subseteq f_1$ and
- $b_0 \subseteq b_1$.

(This is the natural version of Baumgartner's forcing for adding a club with finite conditions incorporating promises to avoid relevant C_δ's.)

$\mathcal{P}_{\vec{C}}$ is proper:

Let $(f, b) \in N$, where $N \prec H(\theta)$ for quite large θ.
Let $\delta_N = N \cap \omega_1 \in \omega_1$. Then $(f \cup \{(\delta_N, \delta_N)\}, b)$ is $(N, \mathcal{P}_{\vec{C}})$–generic:

Let (f', b') extend $(f \cup \{(\delta_N, \delta_N)\}, b)$ and let $D \subseteq \mathcal{P}$ dense and open, $D \in N$. By extending (f', b') if necessary we may assume $(f', b') \in D$.

Note: $f' \restriction \delta_N$, $b' \restriction \delta_N \in N$. In N pick θ_0 large enough and let $(M_\nu)_{\nu < \omega_1}$ $\subseteq$–continuous chain of countable elementary substructures of $H(\theta_0)$ containing $f' \restriction \delta_N$, $b' \restriction \delta_N$ and D.

$(\delta_{M_\nu})_{\nu<\delta_N}$ is a club of δ_N of order type δ_N. Hence we may find ν such that $\delta_{M_\nu} \notin C_{\delta_N}$ and $\delta_{M_\nu} \notin C_{f'(\delta)}$ for any $\delta \in \mathrm{dom}(f')$ above δ_N. There is also $\eta < \delta_{M_\nu}$ such that $[\eta, \delta_{M_\nu}) \cap C_{\delta_N} = \emptyset$ and $[\eta, \delta_{M_\nu}) \cap C_{f'(\delta)} = \emptyset$ for any $\delta \in \mathrm{dom}(f')$ above δ_N.

Now work inside M_ν. By correctness, there is, in M_ν, a condition $(\bar{f}, \bar{b}) \in D$ extending $(f' \restriction \delta_N, b' \restriction \delta_N)$ and such that $\min(\bar{f} \setminus (f' \restriction \delta_n)) > \eta$ (as witnessed by (f', b') itself!).

Finally, $(f' \cup \bar{f}, b' \cup \bar{b})$ is a $\mathcal{P}_{\vec{C}}$–condition extending both (f', b') and $(\bar{f}, \bar{b})$. $\square$

Remark: In above proof, going from (f, b) to $(f \cup \{(\delta_N, \delta_N)\}, b)$ can be seen as implicitly "adding N as side condition".

Note: It follows from the above that PFA implies $\neg$ WCG.

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Remark: In above proof, going from (f, b) to $(f \cup \{(\delta_N, \delta_N)\}, b)$ can be seen as implicitly "adding N as side condition".

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Measuring is the following statement: Suppose $\vec{C} = (C_\delta : \delta \in \mathrm{Lim}(\omega_1))$ is such that each C_δ is a closed subset of δ in the order topology. Then there is a club $D \subseteq \omega_1$ such that for every $\delta \in D$ there is some $\alpha < \delta$ such that either

- $(D \cap \delta) \setminus \alpha \subseteq C_\delta$, or else
- $(D \setminus \alpha) \cap C_\delta = \emptyset$.

We say that D *measures* $\vec{C}$.

• Measuring is equivalent to Measuring restricted to club–sequences.

• Measuring implies $\neg$ WCG: Let $\vec{C} = (C_\delta : \delta \in \mathrm{Lim}(\omega_1))$ be a ladder system. Let D be a club measuring $\vec{C}$. Then D' is such that each $\delta \in D'$ has finite intersection with C_δ. Indeed, we can assume that δ is a limit point of D'. But then $D \cap \delta$ cannot have a tail contained in C_δ since it is a limit point of limit points of D and $\mathrm{ot}(C_\delta) = \omega$. Hence $D \cap \delta$ has a tail disjoint from C_δ.

Given a set of ordinals X and an ordinal α say that

- $\mathrm{rank}(X, \alpha) > 0$ iff α is a limit point of ordinals in X, and
- if $\rho > 1$, then $\mathrm{rank}(X, \alpha) \geq \rho$ iff for every $\rho' < \rho$, α is a limit point of ordinals β such that $\mathrm{rank}(X, \beta) \geq \rho'$.

Measuring one club–sequence with finite conditions:

Let $\vec{C} = (C_\delta \,:\, \delta \in \mathrm{Lim}(\omega_1))$ club–sequence. Let $\mathcal{P}_{\vec{C}}$ be as follows: Conditions are triples $(f, b, \mathcal{N})$ such that

(1) $f \subseteq \omega_1 \times \mathrm{Lim}(\omega_1)$ is a finite function.

(2) $\mathrm{dom}(b) \subseteq \mathrm{dom}(f)$ and $b(\xi) < f(\xi)$ for each $\xi \in \mathrm{dom}(b)$.

(3) For each $\xi \in \mathrm{dom}(b)$, $C_{f(\xi)} \cap \mathrm{range}(f \restriction \xi) \subseteq b(\xi)$.

(4) $\mathcal{N}$ is a finite $\in$–chain of countable elementary submodels of $H(\omega_2)$.

(5) The following holds for every $\nu \in \mathrm{dom}(f)$.

  (5.1) For every $N \in \mathcal{N}$ such that $\delta_N \leq f(\nu)$ and every club $C \subseteq \omega_1$ in N, $\mathrm{rank}(C, f(\nu)) \geq \nu$.

  (5.2) If $\nu \in \mathrm{dom}(b)$, then for every $N \in \mathcal{N}$ such that $\delta_N \leq f(\nu)$ and every club $C \subseteq \omega_1$ in N, $\mathrm{rank}(C \setminus C_{f(\nu)}, f(\nu)) \geq \nu$.

(6) For every $N \in \mathcal{N}$, $(\delta_N, \delta_N) \in f$.

Extension: $(f_1, b_1, \mathcal{N}_1) \leq (f_0, b_0, \mathcal{N}_0)$ iff

- $f_0 \subseteq f_1$,
- $b_0 \subseteq b_1$, and
- $\mathcal{N}_0 \subseteq \mathcal{N}_1$.

$\mathcal{P}_{\vec{C}}$ is proper:

Let $(f, b, \mathcal{N}) \in N$, where $N \prec H(\theta)$ for quite large θ. Let $\delta_N = N \cap \omega_1 \in \omega_1$. Then $(f \cup \{(\delta_N, \delta_N)\}, b, \mathcal{N} \cup \{N \cap H(\omega_2)\})$ is $(N, \mathcal{P}_{\vec{C}})$–generic:

Let $(f', b', \mathcal{N}')$ extend $(f \cup \{(\delta_N, \delta_N)\}, b, \mathcal{N} \cup \{N \cap H(\omega_2)\})$ and let $D \subseteq \mathcal{P}$ dense and open, $D \in N$. By extending $(f', b', \mathcal{N}')$ if necessary we may assume $(f', b', \mathcal{N}') \in D$.

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Note: $f' \restriction \delta_N$, $b' \restriction \delta_N$, $\mathcal{N}' \cap N \in N$. In N pick θ_0 large enough and let $(M_\nu)_{\nu<\omega_1}$ $\subseteq$–continuous chain of countable elementary substructures of $H(\theta_0)$ containing $f' \restriction \delta_N$, $b' \restriction \delta_N$, $\mathcal{N}' \cap N$ and D. Let $C = (\delta_{M_\nu})_{\nu<\omega_1}$.

Assume $\delta_N \in \mathrm{dom}(b')$ (proof in the other case is easier). But then there is some ν such that $\delta_{M_\nu} \notin C_{\delta_N}$ and $\delta_M \notin C_{f(\delta')}$ for any $\delta' \in \mathrm{dom}(b')$ such that $\delta' > \delta_N$. By closedness of the C_δ's, there is also $\eta < \delta_M$ such that $[\eta, \delta_M) \cap C_{\delta_N} = \emptyset$ and $[\eta, \delta_M) \cap C_{f'(\delta)} = \emptyset$ for any $\delta \in \mathrm{dom}(f')$ above δ_N.

The rest of the proof is now as in the $\neg\,\mathrm{WCG}$ case. $\square$

$\mathcal{P}_{\vec{C}}$ measures $\vec{C}$:

Easy: If G is $\mathcal{P}_{\vec{C}}$–generic and $F_G = \bigcup\{f : (f, b, \mathcal{N}) \in G \text{ for some } b, \mathcal{N}\}$, then $\mathrm{range}(F_G)$ is a club of ω_1 and for each limit ordinal $\delta \in \omega_1$, if $\delta \in \mathrm{dom}(b)$ for some $(f, b, \mathcal{N}) \in G$, then a tail of $\mathrm{range}(F_G)$ is disjoint from $C_{f(\delta)}$.

Now suppose there is no $(f, b, \mathcal{N}) \in G$ such that $\delta \in \mathrm{dom}(b)$. Pick $(f, b, \mathcal{N})$ such that $\delta \in \mathrm{dom}(f)$. We may assume there is $N \in \mathcal{N}$ with $\delta_N \leq \delta$ and a club $C \in N$ such that $\mathrm{rank}(C \setminus C_{f(\delta)}, f(\delta)) = \delta_0 < \delta$. Otherwise we would be able to extend $(f, b, \mathcal{N})$ to $(f, b', \mathcal{N})$ such that $\delta \in \mathrm{dom}(b')$. But then, if $(f', b', \mathcal{N}) \leq (f, b, \mathcal{N})$ and $\delta_0 \in \mathrm{dom}(f')$, $(f', b', \mathcal{N})$ forces that $\mathrm{range}(F_{\dot{G}}) \cap [f'(\delta_0), f(\delta)) \subseteq C_{f(\delta)}$. $\square$

Hence, PFA implies Measuring.

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Hence, PFA implies Measuring.

Iterated forcing with side conditions

Recall our problem: Iterate (interesting) non–c.c.c. proper forcing while getting $2^{\aleph_0} > \aleph_2$ in the end.

Neither countable supports, nor uncountable supports nor finite supports work.

A solution: Use finite supports, together with countable elementary substructures of some $H(\theta)$ as side conditions affecting the whole iteration or initial segments of the iteration in order to ensure properness. As mentioned, the idea of using countable structures as side conditions in order to "force" a non–proper forcing to become proper is old. However, the idea of doing this in the context of actual iterations is relatively new.

Typically we will want our iteration to have the $\aleph_2$–c.c. (after all we are interested in $2^{\aleph_0}$ arbitrarily large).The natural approach of using finite $\in$–chains of structures won't work, though, since we have too many structures and would therefore lose the $\aleph_2$–c.c. We will replace $\in$–chains of structures by "matrices" of structures with suitable symmetry properties. If we start with CH and consider only iterands with the $\aleph_2$–c.c., we may succeed.

Symmetric systems of elementary substructures

Definition

Let θ be a cardinal and $T \subseteq H(\theta)$ (such that $\bigcup T = H(\theta)$). A finite set $\mathcal{N} \subseteq [H(\theta)]^{\aleph_0}$ is a T–*symmetric system* iff the following holds for all N, N_0, $N_1 \in \mathcal{N}$:

(1) $(N; \in, Y) \preccurlyeq (H(\theta); \in, T)$

(2) If $\delta_{N_0} = \delta_{N_1}$, then there is a unique isomorphism

$$\Psi_{N_0,N_1} : (N_0; \in, T) \longrightarrow (N_1; \in, T)$$

Furthermore, Ψ_{N_0,N_1} is the identity on $N_0 \cap N_1$.

(3) If $\delta_{N_0} = \delta_{N_1}$ and $N \in N_0 \cap \mathcal{N}$, then $\Psi_{N_0,N_1}(N) \in \mathcal{N}$.

(4) If $\delta_{N_0} < \delta_{N_1}$, then there is some $N_1' \in \mathcal{N}$ such that $\delta_{N_1'} = \delta_{N_1}$ and $N_0 \in N_1'$.

- Symmetric systems had previously been considered in (at least) work of Todorčević, Abraham–Cummings and Koszmider. Again, not in the context of forcing iterations.
- The def. of symmetric system guarantees that

(4)' if $N_0, N_1 \in \mathcal{N}$ and $\delta_{N_0} < \delta_{N_1}$, then there is some $N_0' \in N_1 \cap \mathcal{N}$ such that $\delta_{N_0'} = \delta_{N_0}$ and $N_0 \cap N_1 = N_0 \cap N_0'$.

(In fact, $N_0' = \Psi_{N_1', N_1}(N_0)$, where $N_1' \in \mathcal{N}$ is such that $\delta_{N_1'} = \delta_{N_1}$ and $N_0 \in N_1'$.) This property is important in many applications. Sometimes it is enough to keep (1)–(3) and weaken (4) to (4)'. The resulting object is called *partial T–symmetric system*.

Two amalgamation lemmas

1st amalgamation lemma: If $\mathcal{N}$ and $\mathcal{N}'$ are T–symmetric systems, $(\bigcup\mathcal{N}) \cap (\bigcup\mathcal{N}') = X$, and there are enumerations $(N_i)_{i<n}$ and $(N'_i)_{i<n}$ of $\mathcal{N}$, $\mathcal{N}'$, resp., for which there is an isomorphism

$$\Psi : (\bigcup\mathcal{N}; \in, N_i, T, X)_{i<n} \longrightarrow (\bigcup\mathcal{N}'; \in, N'_i, T, X)_{i<n}$$

then $\mathcal{N} \cup \mathcal{N}'$ is a T–symmetric system.

2nd amalgamation lemma: Let $\mathcal{N}$ be a T–symmetric system and $M \in \mathcal{N}$. Suppose $\mathcal{M} \in M$ is a T–symmetric system such that $\mathcal{N} \cap M \subseteq \mathcal{M}$. Let

$$\mathcal{N}^M(\mathcal{M}) = \mathcal{N} \cup \{\Psi_{M,M'}(N) \ : \ N \in \mathcal{M},\ M' \in \mathcal{N} \ : \ \delta_{M'} = \delta_M\}$$

Then $\mathcal{N}^M(\mathcal{M})$ is a T–symmetric system.

Corollaries Let

$$\mathrm{Symm}_T = (\{\mathcal{N} : \mathcal{N}\ T\text{–symmetric system}\}, \supseteq)$$

Using 1st amalgamation lemma:

Corollary 1 (CH) Symm_T is $\aleph_2$–Knaster.

Using 2nd amalgamation lemma:

Corollary 2 Symm_T is proper.

Using Corollary 2 and the proof of Corollary 1:

Corollary 3 (CH) Symm_T adds new reals but preserves CH.

Iterating: A typical construction.

Start with CH, let κ regular with $2^{<\kappa} = \kappa$. Fix suitable $T \subseteq H(\kappa)$. Let $(\mathcal{P}_\alpha : \alpha \leq \kappa)$ be such that for all α, a condition in $\mathcal{P}_\alpha$ is a pair $q = (F, \Delta)$ such that:

(1) F is a finite function such that $\mathrm{dom}(F) \subseteq \alpha$ ($\mathrm{dom}(F)$ is the support of q).

(2) Δ is a finite set of pairs (N, γ), where $N \in [H(\kappa)]^{\aleph_0}$, $\gamma \leq \alpha$, $\gamma \leq \sup(N \cap \kappa)$, and where $\mathrm{dom}(\Delta)$ is a (partial) T–symmetric system (γ is the *marker* associated to N).

(3) For all $\beta < \alpha$,

$$q|_\beta := (F \restriction \beta, \{(N, \min\{\gamma, \beta\}) : (N, \gamma) \in \Delta\})$$

is a $\mathcal{P}_\beta$–condition.

(4) For every $\xi \in \mathrm{dom}(F)$,

$$q|_\xi \Vdash_{\mathcal{P}_\xi} F(\xi) \in \Phi^*(\xi)$$

where $\Phi^*(\xi)$ is a $\mathcal{P}_\xi$–name for a suitable forcing, and $\Phi^*(\xi) = \Phi(\xi)$ if $\Phi(\xi)$ is a $\mathcal{P}_\xi$–name for a suitable forcing (and where Φ is a suitable bookkeeping function on κ).

(5) For every $\xi \in \mathrm{dom}(F)$ and every $(N, \gamma) \in \Delta$, if $\xi \leq \gamma$ and $\xi \in N$, then

$$q|_\xi \Vdash_{\mathcal{P}_\xi} F(\xi) \text{ is } (N[\dot{G}_\xi], \Phi^*(\xi))\text{–generic}$$

Given $\mathcal{P}_\alpha$–conditions $q_0 = (F_0, \Delta_0)$, $q_1 = (F_1, \Delta_1)$, $q_1 \leq_\alpha q_0$ iff

(a) for every $(N, \gamma) \in \Delta_0$ there is some $\gamma' \geq \gamma$ such that $(N, \gamma') \in \Delta_1$,

(b) $\mathrm{dom}(F_0) \subseteq \mathrm{dom}(F_1)$, and

(c) for every $\xi \in \mathrm{dom}(F_0)$,

$$q_0|_\xi \Vdash_{\mathcal{P}_\xi} F_1(\xi) \leq_{\Phi^*(\xi)} F_0(\xi)$$

This way it is for example possible to build models of forcing axioms for classes Γ such that $\{\mathbb{P} : \mathbb{P}\ \text{c.c.c.}\} \subseteq \Gamma \subseteq \{\mathbb{P} : \mathbb{P}\ \text{proper}\}$ together with $2^{\aleph_0} > \aleph_2$.

[More of this later.]

Main example for this talk: Measuring together with $2^{\aleph_0} > \aleph_2$

Theorem

(A.–Mota (JSL 2017, to appear)) (CH) Let κ be a cardinal such that $2^{<\kappa} = \kappa$ and $\kappa^{\aleph_1} = \kappa$. There is then a partial order $\mathcal{P}$ with the following properties.

(1) *$\mathcal{P}$ is proper and $\aleph_2$–Knaster.*

(2) *$\mathcal{P}$ forces the following statements.*

- *Measuring*
- *$2^{\mu} = \kappa$ for every infinite cardinal $\mu < \kappa$.*

Proof of the main theorem

Yet another notion of rank: Given sets N, $\mathcal{X}$ and an ordinal η, we define $\mathrm{rank}(\mathcal{X}, N) \geq \eta$ recursively by:

- $\mathrm{rank}(\mathcal{X}, N) \geq 1$ if and only if for every $a \in N$ there is some $M \in \mathcal{X} \cap N$ such that $a \in M$.
- If $\rho > 1$, then $\mathrm{rank}(\mathcal{X}, N) \geq \rho$ if and only if for every $\rho' < \rho$ and every $a \in N$ there is some $M \in \mathcal{X} \cap N$ such that $a \in M$ and $\mathrm{rank}(\mathcal{X}, M) \geq \rho'$.

Let $\Phi : \kappa \longrightarrow H(\kappa)$ be such that $\Phi^{-1}(x)$ is unbounded in κ for all $x \in H(\kappa)$. Notice that Φ exists by $2^{<\kappa} = \kappa$. Let also $\lhd$ be a well–order of $H((2^\kappa)^+)$.

Let $(\theta_\alpha)_{\alpha<\kappa}$ be the sequence of cardinals defined by $\theta_0 = |H((2^\kappa)^+)|^+$ and $\theta_\alpha = (2^{<\sup_{\beta<\alpha}\theta_\beta})^+$ if $\alpha > 0$.

For each $\alpha < \kappa$ let $\mathcal{M}^*_\alpha$ be the collection of all countable elementary substructures of $H(\theta_\alpha)$ containing Φ, $\lhd$ and $(\theta_\beta)_{\beta<\alpha}$, and let

$$\mathcal{M}_\alpha = \{N^* \cap H(\kappa) \,:\, N^* \in \mathcal{M}^*_\alpha\}$$

Let T^α be the $\lhd$–first $T \subseteq H(\kappa)$ such that for every $N \in [H(\kappa)]^{\aleph_0}$, if $(N, \in, T \cap N) \prec (H(\kappa), \in, T)$, then $N \in \mathcal{M}_\alpha$.

Let also

$$\mathcal{T}^\alpha = \{N \in [H(\kappa)]^{\aleph_0} \,:\, (N, \in, T^\alpha \cap N) \prec (H(\kappa), \in, T^\alpha)\}.$$

Fact

Let $\alpha < \beta \leq \kappa$*.*

1. *If* $N^* \in \mathcal{M}^*_\beta$ *and* $\alpha \in N^*$*, then* $\mathcal{M}^*_\alpha \in N^*$ *and* $N^* \cap H(\kappa) \in \mathcal{T}^\alpha$*.*
2. *If* $N, N' \in \mathcal{T}^\beta$*,* $\Psi : (N, \in, T^\beta \cap N) \longrightarrow (N', \in, T^\beta \cap N')$ *is an isomorphism, and* $M \in N \cap \mathcal{T}^\beta$*, then* $\Psi(M) \in \mathcal{T}^\beta$*.*

Our forcing $\mathcal{P}$ will be $\mathcal{P}_\kappa$, where $(\mathcal{P}_\beta : \beta \leq \kappa)$ is the sequence of posets to be defined next.

In the following definition, and throughout the lectures, if q is an ordered pair (F, Δ), we will denote F and Δ by F_q and Δ_q, respectively.

Let $\beta \leq \kappa$ and suppose $\mathcal{P}_\alpha$ has been defined for all $\alpha < \beta$. Conditions in $\mathcal{P}_\beta$ are ordered pairs $q = (F, \Delta)$ with the following properties.

(1) F is a finite function with $\mathrm{dom}(F) \subseteq \beta$.

(2) Δ is a finite set of pairs (N, γ) such that $N \in [H(\kappa)]^{\aleph_0}$ and γ is an ordinal such that $\gamma \leq \beta$ and $\gamma \leq \sup(N \cap \kappa)$.

(3) $\mathcal{N}^q_\beta := \{N : (N, \beta) \in \Delta,\ \beta \in N\}$ is a T^β–symmetric system.

(4) For every $\alpha < \beta$, the restriction of q to α,

$$q|_\alpha := (F \restriction \alpha, \{(N, \min\{\alpha, \gamma\}) : (N, \gamma) \in \Delta\}),$$

is a condition in $\mathcal{P}_\alpha$.

(5) Suppose $\beta = \alpha + 1$. Let $\mathcal{N}^{\dot{G}_\alpha}$ be a $\mathcal{P}_\alpha$–name for $\bigcup\{\mathcal{N}^r_\alpha : r \in \dot{G}_\alpha\}$ (where $\dot{G}_\alpha$ is the canonical $\mathcal{P}_\alpha$–name for the generic object). Let $\dot{C}^\alpha$ be a $\mathcal{P}_\alpha$–name for a club–sequence on ω_1 such that $\mathcal{P}_\alpha$ forces that

- $\dot{C}^\alpha = \Phi(\alpha)$ in case $\Phi(\alpha)$ is a $\mathcal{P}_\alpha$–name for a club–sequence on ω_1, and that
- $\dot{C}^\alpha$ is some fixed club–sequence on ω_1 in the other case.

If $\alpha \in \mathrm{dom}(F)$, then $F(\alpha) = (f, b, \mathcal{O})$ has the following properties.

(a) $f \subseteq \omega_1 \times \omega_1$ is a finite strictly increasing function.

(b) $\mathcal{O} \subseteq \mathcal{N}^{q|\alpha}_\alpha$ is a T^β–symmetric system.

(c) $\mathrm{range}(f) \subseteq \{\delta_N : N \in \mathcal{O}\}$

(d) For every $\delta \in \mathrm{dom}(f)$, if $N \in \mathcal{O}$ is such that $p(\delta) = \delta_N$, then

$$q|_\alpha \Vdash_{\mathcal{P}_\alpha} \mathrm{rank}(\mathcal{N}^{\dot{G}_\alpha} \cap \mathcal{T}^\beta, N) \geq \delta$$

(e) $\mathrm{dom}(b) \subseteq \mathrm{dom}(f)$ and $b(\delta) < f(\delta)$ for every $\delta \in \mathrm{dom}(b)$.

(f) For every $\delta \in \mathrm{dom}(b)$,

$$q|_\alpha \Vdash_{\mathcal{P}_\alpha} \mathrm{range}(f \restriction \delta) \cap \dot{C}^\alpha(f(\delta)) \subseteq b(\delta)$$

(g) For every $\delta \in \mathrm{dom}(b)$, if $N \in \mathcal{O}$ is such that $f(\delta) = \delta_N$, then

$$q|_\alpha \Vdash_{\mathcal{P}_\alpha} \mathrm{rank}(\{M \in \mathcal{N}^{\dot{G}_\alpha} \cap \mathcal{T}^\beta : \delta_M \notin \dot{C}^\alpha(f(\delta))\}, N) \geq \delta$$

(h) If $N \in \mathcal{N}^q_\beta$, then $N \in \mathcal{O}$, $\delta_N \in \mathrm{dom}(f)$ and $f(\delta_N) = \delta_N$.

Given $\mathcal{P}_\beta$–conditions $q_i = (F_i, \Delta_i)$, for $i = 0$, 1, q_1 extends q_0 if and only if

- $\text{dom}(F_0) \subseteq \text{dom}(F_1)$ and for all $\alpha \in \text{dom}(F_0)$, if $F_0(\alpha) = (f, b, \mathcal{O})$ and $F_1(\alpha) = (f', b', \mathcal{O}')$, then $f \subseteq f'$, $b \subseteq b'$ and $\mathcal{O} \subseteq \mathcal{O}'$, and
- $\Delta_0 \subseteq \Delta_1$